AP Calculus 12 **1.4 Trigonometric Limits** Name______________

Date_______________

A few limit properties we will use:

$$\lim_{x\to a}\big(f(x)+g(x)\big) = \lim_{x\to a} f(x) + \lim_{x\to a} g(x)$$

$$\lim_{x\to a}\big(f(x)g(x)\big) = \left(\lim_{x\to a} f(x)\right)\left(\lim_{x\to a} g(x)\right)$$

$$\lim_{x\to a}\big(f(x)\big)^n = \left(\lim_{x\to a} f(x)\right)^n$$

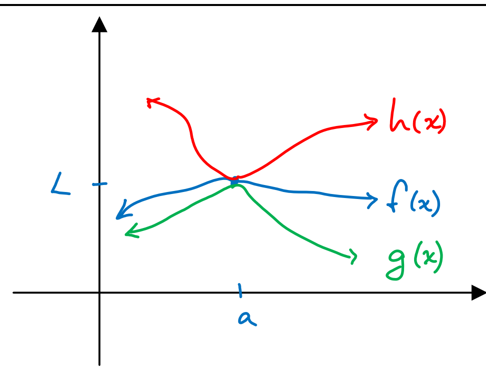
The Squeeze Theorem:

If g(x) ≤ f(x) ≤ h(x), and

$\lim_{x\to a} g(x) = L$ and $\lim_{x\to a} h(x) = L$, then

$$\lim_{x\to a} f(x) = L.$$

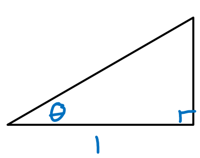
i.e.



This will be used for 2 specific limits:

$\lim_{x\to 0}\frac{\sin x}{x} = 1$ and $\lim_{x\to 0}\frac{1-\cos x}{x} = 0$

Proof: by areas



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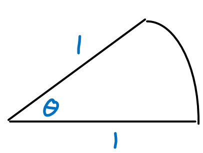

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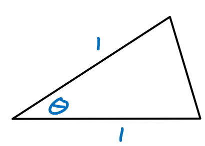

$A = \frac{b \times h}{2}$

$= \frac{1 \times \tan\theta}{2}$

$A = \pi(1)^2 \times \frac{\theta}{2\pi}$

$= \frac{\theta}{2}$

$A = \frac{ab\sin C}{2}$

$= \frac{1\cdot 1\cdot \sin\theta}{2}$

$$\frac{\tan\theta}{2} \geq \frac{\theta}{2} \geq \frac{\sin\theta}{2}$$

multiply by 2:

$$\tan\theta \geq \theta \geq \sin\theta$$

$$\frac{\sin\theta}{\cos\theta} \geq \theta \geq \sin\theta$$

Divide all by $\sin\theta$

$$\frac{1}{\cos\theta} \geq \frac{\theta}{\sin\theta} \geq 1$$

Take reciprocals of each.....

$$\cos\theta \leq \frac{\sin\theta}{\theta} \leq 1$$

So....

$$\lim_{\theta\to 0} \cos\theta \leq \lim_{\theta\to 0} \frac{\sin\theta}{\theta} \leq \lim_{\theta\to 0} 1$$

$$1 \leq \lim_{\theta\to 0} \frac{\sin\theta}{\theta} \leq 1$$

$\therefore$ by the Squeeze thm .....

$$\boxed{\lim_{\theta\to 0} \frac{\sin\theta}{\theta} = 1}$$

Examples:

a) $\lim_{x\to 0}\frac{\sin x}{7x} = \lim_{x\to 0} \frac{1}{7}\cdot\frac{\sin x}{x}$

$= \lim_{x\to 0}\frac{1}{7}\cdot\lim_{x\to 0}\frac{\sin x}{x}$

$= \frac{1}{7}\cdot 1 \longrightarrow \frac{1}{7}$

b) $\lim_{x\to 0}\frac{\sin(3x)}{x} = \lim_{x\to 0}\frac{3\sin(3x)}{3x}$

$= \lim_{x\to 0} 3\cdot\lim_{x\to 0}\frac{\sin(3x)}{3x}$

as $x\to$, $3x\to 0$ too, so you can treat it as its own variable.

$= 3\cdot 1 \longrightarrow 3$

c) $\lim_{x\to 0}\frac{\sin^2 x}{x} =$

2 ways

$= \lim_{x\to 0}\frac{\sin x\cdot\sin x}{x}$

$= \lim_{x\to 0}\sin x\cdot\lim_{x\to 0}\frac{\sin x}{x}$

$= 0\cdot 1$

$= 0$

$\lim_{x\to 0}\frac{\sin^2 x}{x} = \lim_{x\to 0}\frac{1-\cos^2 x}{x}$

$= \lim_{x\to 0}\frac{(1-\cos x)(1+\cos x)}{x}$

$= \lim_{x\to 0}\frac{1-\cos x}{x}\cdot\lim_{x\to 0} 1+\cos x$

$= (0)\cdot(1+1)$

$= 0$

Assignment: p. 75# 29-31 p. 82# 29-39 odds

P. 75 # 29 → 31

(29)

a) $\lim_{x \to c} [5\, g(x)]$

$= 5 \lim_{x \to c} g(x)$

$= 5(3) = \underline{15}$

b) $\lim_{x \to c} [f(x) + g(x)]$

$= \lim_{x \to c} f(x) + \lim_{x \to c} g(x)$

$= 2 + 3 = \underline{5}$

c) $\lim_{x \to c} [f(x)\, g(x)]$

$= \left(\lim_{x \to c} f(x)\right)\left(\lim_{x \to c} g(x)\right)$

$= (2)(3) \longrightarrow \underline{6}$

d) $\lim_{x \to c} \frac{f(x)}{g(x)}$

$= \dfrac{\lim_{x \to c} f(x)}{\lim_{x \to c} g(x)} = \underline{\dfrac{2}{3}}$

(30)

a) $\lim_{x \to c} [4 f(x)]$

$= 4 \lim_{x \to c} f(x)$

$= 4\left(\frac{3}{2}\right) = \underline{6}$

b) $\lim_{x \to c} [f(x) + g(x)]$

$= \lim_{x \to c} f(x) + \lim_{x \to c} g(x)$

$= \frac{3}{2} + \frac{1}{2} \rightarrow \frac{4}{2} = \underline{2}$

c) $\lim_{x \to c} (f(x)g(x))$

$= \left(\lim_{x \to c} f(x)\right)\left(\lim_{x \to c} g(x)\right)$

$= \left(\frac{3}{2}\right)\left(\frac{1}{2}\right) \longrightarrow \underline{3/4}$

d) $\lim_{x \to c} \frac{f(x)}{g(x)}$

$= \dfrac{\lim_{x \to c} f(x)}{\lim_{x \to c} g(x)} = \dfrac{\frac{3}{2}}{\frac{1}{2}} = \frac{3}{2} \times \frac{2}{1} \rightarrow \underline{3}$

(31)

a) $\lim_{x \to c} [f(x)]^3$

$= \left[\lim_{x \to c} f(x)\right]^3$

$= 4^3 \longrightarrow \underline{64}$

b) $\lim_{x \to c} \sqrt{f(x)}$

$= \sqrt{\lim_{x \to c} f(x)}$

$= \sqrt{4} \longrightarrow \underline{2}$

c) $\lim_{x \to 3} [3 f(x)]$

$= 3 \lim_{x \to 3} f(x)$

$= 3(4) \longrightarrow \underline{12}$

d) $\lim_{x \to c} [f(x)]^{3/2}$

$= \left[\lim_{x \to c} f(x)\right]^{3/2}$

$= [4]^{3/2}$

$= \left(\sqrt{4}\right)^3 = 2^3 = \underline{8}$

P. 82 # 29 → 39.

(29) $\lim_{x\to 0} \frac{\sin^2 x}{x}$

$= \lim_{x\to 0} \sin x \left(\frac{\sin x}{x}\right)$

$= \sin(0)(1) = \underline{\underline{0}}$

(30) $\lim_{x\to 0} \frac{\tan^2 x}{x}$

$= \lim_{x\to 0} \frac{\sin^2 x}{\cos^2 x} \cdot \frac{1}{x}$

$= \lim_{x\to 0} \frac{\sin x}{x} \cdot \frac{\sin x}{\cos^2 x}$

$= (1)\left(\frac{\sin 0}{\cos^2 0}\right)$

$= 1\left(\frac{0}{1}\right) \longrightarrow \underline{\underline{0}}$

(31) $\lim_{x\to 0} \frac{(1-\cos x)^2}{x}$

$= \lim_{x\to 0} \frac{1-\cos x}{x}(1-\cos x)$

$= (0)(1-\cos 0)$

$= 0\,(0) \longrightarrow \underline{\underline{0}}$

(32) $\lim_{\phi\to\pi} \phi \sec\phi$

$= \lim_{\phi\to\pi} \frac{\phi}{\cos\phi}$

$= \frac{\pi}{\cos\pi} = \frac{\pi}{-1} = \underline{\underline{-\pi}}$

(34) $\lim_{x\to\frac{\pi}{4}} \frac{1-\tan x}{\sin x - \cos x}$

$= \lim_{x\to\frac{\pi}{4}} \frac{1-\frac{\sin x}{\cos x}}{\sin x - \cos x}\left(\frac{\cos x}{\cos x}\right)$

$= \lim_{x\to\frac{\pi}{4}} \frac{\cos x - \sin x}{(\sin x - \cos x)(\cos x)}$

$= \lim_{x\to\frac{\pi}{4}} \frac{-1}{\cos x} \longrightarrow \frac{-1}{\cos\frac{\pi}{4}}$

$= \frac{-1}{\frac{\sqrt{2}}{2}} \longrightarrow \underline{\underline{-\sqrt{2}}}$

(38) $\lim_{x\to\frac{\pi}{2}} \frac{\cos x}{\cot x}$

$= \lim_{x\to\frac{\pi}{2}} \frac{\cos x}{\frac{\cos x}{\sin x}}$

$= \lim_{x\to\frac{\pi}{2}} \cos x \cdot \frac{\sin x}{\cos x}$

$= \lim_{x\to\frac{\pi}{2}} \sin x$

$= \sin\frac{\pi}{2} \longrightarrow \underline{\underline{1}}$

(35) $\lim_{t \to 0} \frac{\sin^2 t}{t^2}$

$= \lim_{t \to 0} \left(\frac{\sin t}{t}\right)^2$

$= (1)^2 \to \boxed{1}$

(36) $\lim_{x \to 0} \frac{\sin 2x}{\sin 3x}$

$= \lim_{x \to 0} \sin 2x \cdot \frac{1}{\sin 3x}$

$= \lim_{x \to 0} \frac{2x \sin(2x)}{2x} \cdot \frac{3x}{3x \sin(3x)}$

$= \lim_{x \to 0} \frac{\sin(2x)}{2x} \cdot \frac{3x}{\sin(3x)} \cdot \frac{2}{3}$

$= \lim_{x \to 0} \left(\frac{\sin 2x}{2x}\right)\left(\frac{\sin 3x}{3x}\right)^{-1} \cdot \frac{2}{3}$

$= (1)(1)^{-1} \cdot \frac{2}{3} \to \underline{\underline{\frac{2}{3}}}$

(37) $\lim_{t \to 0} \frac{\sin 3t}{t}$

$= \lim_{t \to 0} \frac{3 \sin 3t}{3t}$

$= \lim_{t \to 0} 3 \cdot \frac{\sin 3t}{3t}$

$= 3(1) \longrightarrow \underline{3}$

(38) $\lim_{h \to 0} (1 + \cos 2h)$

$= 1 + \cos(0)$

$= 1 + 1 \longrightarrow \underline{\underline{2}}$

(39) $\lim_{x \to 0} \frac{\sin(x^2)}{x}$

$= \lim_{x \to 0} \frac{x \sin(x^2)}{x^2}$

$= \lim_{x \to 0} x \cdot \frac{\sin x^2}{x^2}$

$= (0)(1) \longrightarrow \underline{\underline{0}}$